J. Appl. Maths Mechs, Vol. 55, No. 2, pp. 168-176, 1991
Printed in Great Britain

0021-8928/91 $15.00+0.00

PROGRAMMED CONSTRUCTIONS IN CONTROL PROBLEMS WITH VECTOR CRITERION*

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The control problem with a vector criterion is considered. The system is noisy and feedback control is used. The motion of a vector optimal guaranteed result (OGR) is introduced. With a scalar criterion, the problem reduces to the traditional problem of optimal guaranteed control, which can be solved by methods of differential game theory /1-4/. An important part of this theory is the extremal aiming method, which, subject to regularity conditions /1-6/, reduces the construction of a positional strategy to solving auxiliary programmed control problems. It is shown that programmed absorption constructions may be applied to control problems with a vector criterion.

We define the optimal guranteed result (OGR) vector multivalued function (OGR VMF) which is an analogue of the OGR function in the control problem with a scalar criterion. Some properties of this function are stated. In particular, we consider the scalarization of OGR VMF. The infinitesimal form /7-10/ of the u-stability property of the OGR VMF is given. Differential inequalities expressing the u-stability property in vector form are used to analyse programmed constructions in the linear control problem with a convex vector criterion. We define the programmed maximin VMF (PM VMF) in this problem and give the regularity conditions when PM VMF and OGR VMF are equal. These regularity conditions essentially ensure the property of u-stability of PM VMF and are obtained from the corresponding infinitesimal inequalities. The regularity conditions in their final form do not contain elements of infinitesimal construction (directional derivative of PM VMF); they only include the main elements of the problem, such as the vector functional and the Hamiltonian of the controlled system. An example demonstrating the efficiency of these conditions is examined.

1. Vector optimal guaranteed result. Consider a controlled system whose dynamics are described by the equation

$$x^{\cdot} = f(t, x, u, v),\ t \in [t_0, \vartheta] = T,\ x \in R^n,\ u \in P \subset R^p, \quad v \in Q \subset R^q \quad f\colon T \times R^n \times P \times Q \to R^n \tag{1.1}$$

Here x is the n-dimensional phase vector, u is the p-dimensional control vector from the compact set P and v is the q-dimensional noise vector, with values in the compact set Q.

We assume that the function f is continuous in all the variables, satisfies the Lipschitz condition on x, and allows continuation of the solutions.

A vector performance functional is defined on the trajectories $x(\cdot)$ of system (1.1):

$$J(x(\cdot)) = \sigma(x(\vartheta)) = (\sigma_1(x(\vartheta)), \ldots, \sigma_m(x(\vartheta))) \tag{1.2}$$

Here $\sigma_i\colon R^n \to R^1$ are continuous functions.

We assume that the control u is generated positionally /3/. The goal is to "minimize" the functional (1.2) in the sense defined below.

We define an order relation on m-dimensional vectors. For $a = (a_1, \ldots, a_m)$, $b = (b_1, \ldots, b_m)$ we take

$$a \leqslant b, \text{ if } a_i \leqslant b_i, \text{ for all } i,$$
$$a \nleqslant b, \text{ if } a_j > b_j \quad \text{for some } j.$$

Here and below, $i, j = 1, \ldots, m$.

We will now define the vector guaranteed result. Let $U\colon T \times R^n \to P$ be a positional strategy, $X(t_*, x_*, U)$ the bundle of trajectories generated by the strategy U from the initial position $(t_*, x_*) \in T \times R^n$.

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*Prikl.Matem.Mekhan.,55,2,212-221,1991

Let

$$\Sigma(t_*, x_*, U) = \{s \in R^m: s = \sigma(x(\vartheta)),\ x(\cdot) \in X(t_*, x_*, U)\} \tag{1.3}$$
$$\Sigma_{\max}(t_*, x_*, U) = \{s^0 \in R^m: s \leqslant s^0 \text{ for all } s \in \Sigma(t_*, x_*, U)\}$$
$$\Sigma_{\max}(t_*, x_*) = \bigcup_U \Sigma_{\max}(t_*, x_*, U)$$
$$\Sigma_{\max} = \{(t, x, s) \in T \times R^n \times R^m: s \in \Sigma_{\max}(t, x)\}$$

Thus, $\Sigma_{\max}(t_*, x_*, U)$ is the set of vector results guaranteed by the strategy U at the point (t_*, x_*) by all the components simultaneously; $\Sigma_{\max}(t_*, x_*)$ is the set of all guaranteed results in position (t_*, x_*).

Definition 1. An optimal guaranteed result (OGR) at the point (t_*, x_*) is the set $c(t_*, x_*)$ of Pareto-minimal points from the set of guaranteed results $\Sigma_{\max}(t_*, x_*)$, i.e.,

$$c(t_*, x_*) = \{s^0 \in \Sigma_{\max}(t_*, x_*): s \not\leqslant s^0 \text{ for all } s \in \Sigma_{\max}(t_*, x_*) \setminus \{s^0\}\} \tag{1.4}$$

Thus, by definition, for the guaranteed vector $s^0 \in c(t_*, x_*)$, $(t_*, x_*) \in T \times R^n$, there exists a positional strategy $U: T \times R^n \to P$ such that $\sigma(x(\vartheta)) \leqslant s^0$ for all $x(\cdot) \in X(t_*, x_*, U)$ and s^0 is Pareto-"best" among all vectors with this property.

Note that Definition 1 is similar to the definition of the optimal guaranteed value in /11/, where the multicriterion guaranteed control problem was analysed in the framework of Pontryagin's first direct method.

The optimal guaranteed result vector multivalued function (OGR VMF) of problem (1.1), (1.2) is the multivalued mapping $(t, x) \to c(t, x): T \times R^n \to 2^{R^m}$.

2. ***Properties of OGR VMF.*** To investigate the properties of the OGR VMF, we will use the auxiliary system

$$x^{\cdot} = f(t, x, u, v),\ t \in T,\ x \in R^n,\ u \in P,\ v \in Q \tag{2.1}$$
$$s_i^{\cdot} = 0$$

Consider the following problem of guiding the motion to the goal set M, which is the epigraph of the vector function σ,

$$M = \{(x, s) \in R^n \times R^m: \sigma(x) \leqslant s\} \tag{2.2}$$

in time ϑ.

For the position $(t_*, x_*, s_*) \in T \times R^n \times R^m$ we need to determine the positional strategy $U_r: T \times R^n \times R^m \to P$ which guides the motion of the system (2.1) to the goal set M in a time ϑ regardless of noise.

Let $W_u \subset T \times R^n \times R^m$ be the positional absorption set in problem (2.1), (2.2), i.e., the set of initial positions (t_*, x_*, s_*) for which the problem of guiding the motions of system (2.1) to the goal M is solvable.

An important property of the OGR VMF $(t, x) \to c(t, x)$ is expressed by the following theorem.

Theorem 1. The epigraph of the OGR VMF (the set $\Sigma_{\max}$ (1.3)) is identical with the positional absorption set W_u of the augmented problem (2.1), (2.2).

The proof of Theorem 1 is based on the alternative theorem /3/* .

Remark 1. It follows from Theorem 1 that the OGR VMF $(t, x) \to c(t, x)$ may be constructed using algorithms and programs designed to solve guaranteed control problems of the form (2.1), (2.2). Such algorithms and programs are being developed, e.g., at the Department of Dynamic Systems, Institute of Mathematics and Mechanics, Ural Division of the Academy of Sciences of the USSR /12/.

Let us state some properties of the OGR VMF. The proof is simple and therefore omitted.

Property 1. Let $\omega_i: T \times R^n \to R^1$ be the OGR function in the control problem for system (1.1) with the scalar criterion

$$J(x(\cdot)) = \sigma_i(x(\vartheta)) \tag{2.3}$$

i.e.,

$$\omega_i(t, x) = \min_U \max_{x(\cdot) \in X(t, x, U)} \sigma_i(x(\vartheta)),\ (t, x) \in T \times R^n$$

*Complete proofs of this and other propositions are given in Taras'yev A.M., Differential Games with Vector Criterion, Sverdlovsk, 1989. Unpublished manuscript, VINITI 11.07.89, 4608-V89.

For any position $(t_*, x_*) \in T \times R^n$ and any vector $s \in c(t_*, x_*)$ we have the vector inequality $\omega_0(t_*, x_*) \leqslant s$, $\omega_0(t_*, x_*) = (\omega_1(t_*, x_*), \ldots, \omega_m(t_*, x_*))$.

Property 2. For any position $(t_*, x_*) \in T \times R^n$ and index i, there exist numbers s_j°, $j \neq i$, such that the vector $s^\circ = (s_1^\circ, \ldots, s_{i-1}^\circ, \omega_i(t_*, x_*), s_{i+1}^\circ, \ldots, s_m^\circ)$ is the OGR, i.e., $s^\circ \in c(t_*, x_*)$.

Property 3. For all positions $(t_*, x_*) \in T \times R^n$ the set $c(t_*, x_*)$ is bounded.

Property 4. The epigraph of the OGR VMF $(t, x) \to c(t, x)$ - the set W_u - is a closed set.

3. *Scalarization of the vector criterion.* Consider the scalarization of the vector criterion. We will use the scalarization technique of /13/. Assume that the OGR VMF has been constructed. Let

$$(t_*, x_*) \in T \times R^n,\ s^\circ = (s_1^\circ, \ldots, s_m^\circ) \in c(t_*, x_*),\ s_i^\circ > 0$$

$$\alpha_i^\circ = (s_i^\circ)^{-1} \Big(\sum_{j=1}^{m} (s_j^\circ)^{-1}\Big)^{-1}$$

We denote by $U^\circ\colon T \times R^n \to P$ the optimal strategy in problem (1.1), (1.2), i.e., $\sigma(x(\vartheta)) \leqslant s^\circ$ for all $x(\cdot) \in X(t_*, x_*, U^\circ)$.

Assume that a scalarized payoff functional is defined for system (1.1) as a scalarization of the vector functional (1.2)

$$J(x(\cdot)) = \max\{\alpha_i^\circ \sigma_i(x(\vartheta))\} \tag{3.1}$$

We denote by $\omega^\circ\colon T \times R^n \to R^1$ the OGR function and by $U^*\colon T \times R^n \to P$ the optimal strategy in the scalar problem (1.1), (3.1), i.e.,

$$\omega^\circ(t_*, x_*) = \min_{U} \max_{x(\cdot) \in X(t_*, x_*, U)} \max_i \{\alpha_i^\circ \sigma_i(x(\vartheta))\} = \max_{x(\cdot) \in X(t_*, x_*, U^*)} \max_i \{\alpha_i^\circ \sigma_i(x(\vartheta))\} \tag{3.2}$$

Theorem 2. The optimal strategy U° of vector problem (1.1), (1.2) is optimal in the scalar problem (1.1), (3.1), i.e.,

$$\omega^\circ(t_*, x_*) = \max_{x(\cdot) \in X(t_*, x_*, U^\circ)} \max_i \{\alpha_i^\circ \sigma_i(x(\vartheta))\}$$

The optimal strategy U^* of the scalar problem (1.1), (3.1) is optimal in the vector problem (1.1), (1.2), i.e.,

$$\sigma(x(\vartheta)) \leqslant s^\circ \text{ for all } x(\cdot) \in X(t_*, x_*, U^*)$$

Also

$$\omega^\circ(t_*, x_*) = \Big(\sum_{j=1}^{m} (s_j^\circ)^{-1}\Big)^{-1},\ s^\circ = (s_1^\circ, \ldots, s_m^\circ) \in c(t_*, x_*)$$

The proof of Theorem 2 is along the same lines as in /13/.

Remark 2. Scalarization of a vector criterion is impossible without a knowledge of the value $s_* \in c(t_*, x_*)$ of the OGR VMF. Moreover, the scalarization coefficients α_i° depend both on the position $(t_*, x_*) \in T \times R^n$ and on the vector value $s_* \in c(t_*, x_*)$ of the OGR VMF. Prior choice of the scalarization coefficients does not allow for the specific features of the problem, such as the dynamics of the controlled system and the structure of the vector performance functional. The solution of the system of scalar problems (1.1), (3.1) obtained by varying the scalarization coefficients $\alpha_i \geqslant 0$, $\alpha_1 + \ldots + \alpha_m = 1$, is computationally costly.

4. *The u-stability property and vector differential inequalities.* One of the main properties of the positional absorption set W_u is the u-stability property: it is the basis of the successive construction algorithms for the set W_u; this property ensures that the programmed absorption set is identical with the positional absorption set, etc. The u-stability property can be extended to vector multivalued functions, including OGR VMF $(t, x) \to c(t, x)$.

We introduce the following notation. We denote by SC the class of VMF $(t, x) \to \omega(t, x)$ that satisfy the following conditions:

1) for all $(t, x) \in T \times R^n$ the set $\omega(t, x) \subset R^m$ is bounded;

2) the epigraph

$$W = \operatorname{epi} \omega = \{(t, x, s) \in T \times R^n \times R^m\colon s^0 \leqslant s,\ s^0 \in \omega(t, x)\} \tag{4.1}$$

is a closed set;

3) for all $(t, x) \in T \times R^n$, $s^{(1)}, s^{(2)} \in \omega(t, x)$ $(s^{(1)} \neq s^{(2)})$ we have $s^{(1)} \nleqslant s^{(2)}$.

Suppose

$$F(\tau, y, l) = \Pi(\tau, y, l) \cap G(\tau, y) \tag{4.2}$$

$$G(\tau, y) = \text{co}\,\{f \in R^m: f = f(\tau, y, u, v),\ u \in P,\ v \in Q\}$$

$$\Pi(\tau, y, l) = \{r \in R^n: \langle l, r\rangle \geqslant H(\tau, y, l)\} \tag{4.3}$$

$$H(\tau, y, l) = \min_{u \in P} \max_{v \in Q} \langle l, f(\tau, y, u, v)\rangle \tag{4.4}$$

$$(\tau, y, l) \in T \times R^n \times S,\ S = \{r \in R^n: \|r\| = 1\}$$

Definition 2. We say that the function $(t, x) \to \omega(t, x)$: $T \times R^n \to 2^{R^m}$, $\omega \in \mathrm{SC}$, is u-stable if for any positions $(t_*, x_*) \in T \times R^n$ $(t_* < \vartheta)$, any vector $s_* \in \omega(t_*, x_*)$, any moment $t \in (t_*, \vartheta]$ and any vector $l \in S$, there exist a solution $x(\cdot)$ of the differential inclusion

$$x^{\cdot}(\tau) \in F(\tau, x(\tau), l),\ x(t_*) = x_*,\ \tau \in [t_*, t]$$

and a vector $s \in \omega(t, x(t))$ such that $s \leqslant s_*$.

Note that Definition 2 is an expression of the u-stability property for the epigraph (4.1) of the function ω. We see that the OGR VMF $(t, x) \to c(t, x)$ is u-stable in the sense of this definition, because its epigraph W_u (the positional absorption set in problem (2.1), (2.2)) is u-stable /3/.

The u-stability property may be defined in different equivalent ways. The infinitesimal form of the u-stability property is particularly useful /7, 9, 10/. The apparatus of derivatives of multivalued mappings may also be used to obtain an equivalent definition of the u-stability property of VMF. Omitting the details, we give the final form of the differential inequalities defining the u-stability property.

Definition 3. The function $\omega \in \mathrm{SC}$ is u-stable if for any positions (t_*, x_*), $t_* \in [t_0, \vartheta) = T^\circ$, $x_* \in R^n$, any vector value $s_* \in \omega(t_*, x_*)$ and any vector $l \in S$ there exists a vector $f \in F(t_*, x_*, l)$ and a vector value of the lower derivative $d \in \partial_-\omega(t_*, x_*, s_*) \mid (f)$ of the function ω such that $d \leqslant 0$.

Here

$$\partial_-\omega(t_*, x_*, s_*) \mid (h) = \{d^\circ \in \nabla\omega(t_*, x_*, s_*) \mid (h): d \nleqslant d^\circ \text{ for all } d \in \nabla\omega(t_*, x_*, s_*) \mid (R) \setminus \{d^\circ\}\} \tag{4.5}$$

$$\nabla\omega(t_*, x_*, s_*) \mid (h) = \{d \in (\bar{R})^m: (h, d) \in D\omega(t_*, x_*, s_*)\} \tag{4.6}$$

$$D\omega(t_*, x_*, s_*) = \{(h, d) \in R^n \times (\bar{R})^m: h = \lim_{k\to\infty}(x_k - x_*)(t_k - t_*)^{-1}$$

$$d = \lim_{k\to\infty}(s_k - s_*)(t_k - t_*)^{-1},\ t_k \to t_*,\ t_k \in (t_*, \vartheta],\ x_k \in R^n,\ s_k \in \omega(t_k, x_k)\} \tag{4.7}$$

$$\bar{R} = R \cup \{+\infty\} \cup \{-\infty\}$$

5. *Programmed Maximin vector multivalued function.* The infinitesimal form of the u-stability property may be used to obtain better regularity conditions for the programmed maximin function in linear control problems with a convex vector criterion.

Let the system dynamics be described by the linear equation

$$x^{\cdot} = B(t)u + C(t)v,\ t \in T,\ x \in R^n,\ u \in P \subset R^p,\ v \in Q \subset R^q \tag{5.1}$$

where $B(t)$ and $C(t)$ are continuous $n \times p$ and $n \times q$ matrices and P and Q are convex compacta.

The payoff functional is defined by

$$J(x(\cdot)) = \varphi(x(\vartheta)) = (\varphi_1(x(\vartheta)), \ldots, \varphi_m(x(\vartheta))) \tag{5.2}$$

where φ_i are convex functions that satisfy the Lipschitz condition.

The epigraph $\Phi = \text{epi}\,\varphi = \{(x, s) \in R^n \times R^m: \varphi(x) \leqslant s\}$ is a convex set. The support function of the epigraph of the function φ will be called the conjugate of the vector function φ. It is defined by the relationship

$$\rho(l, \alpha) = \sup_{x \in R^n} \{\langle l, x\rangle - (\alpha_1\varphi_1(x) + \ldots + \alpha_m\varphi_m(x))\},\ l \in R^n,\ \alpha \in A \tag{5.3}$$

$$A = \{\alpha = (\alpha_1, \ldots, \alpha_m) \in R^m: \alpha_i \geqslant 0,\ \alpha_1 + \ldots + \alpha_m = 1\}$$

where A is an $(m-1)$-dimensional simplex.

Note that the function $l \to \rho(l, \alpha)$: $R^n \to R$ for a fixed $\alpha \in A$ is the conjugate of the function defined as the scalarization $\alpha_1\varphi_1(x) + \ldots + \alpha_m\varphi_m(x)$ of the components φ_i.

We introduce the notation

$Z = \{(t, x, s) \in T \times R^n \times R^m$: for any $v(\cdot)$ there is $u(\cdot)$ such that

$$\varphi(y(\vartheta)) \leqslant s,\ y(\vartheta) = x + \int_t^\vartheta B(\tau)u(\tau)\,d\tau + \int_t^\vartheta C(\tau)v(\tau)\,d\tau\}$$

Here $\tau \to u(\tau)$: $[t, \vartheta] \to P$, $\tau \to v(\tau)$: $[t, \vartheta] \to Q$ are Lebesgue-measurable functions (programmed controls). The set Z is called a programmed absorption set.

From (5.3) it follows that

$$Z = \Big\{(t, x, s) \in T \times R^n \times R^m : \max_{v(\cdot)} \min_{u(\cdot)} \max_{\alpha\in A} \max_{l\in R^n} (\langle l, x\rangle - \langle \alpha, s\rangle + \int_t^\vartheta \langle l, B(\tau)u(\tau)\rangle\,d\tau + \int_t^\vartheta \langle l, C(\tau)v(\tau)\rangle\,d\tau - \rho(l, \alpha)) \leqslant 0\Big\}$$

Transforming the expression in braces, we obtain

$$Z = \{(t, x, s) \in T \times R^n \times R^m : \min_{\alpha\in A} (\langle \alpha, s\rangle - g(t, x, \alpha)) \geqslant 0\} \tag{5.4}$$

$$g(t, x, \alpha) = \max_{l\in \operatorname{dom}\rho(\alpha)} \Big(\langle l, x\rangle + \int_t^\vartheta H(\tau, l)\,d\tau - \rho(l, \alpha)\Big),\ \alpha \in A \tag{5.5}$$

$$H(\tau, l) = \max_{v\in Q} \langle l, C(\tau)v\rangle + \min_{u\in P} \langle l, B(\tau)u\rangle,\ \tau \in [t, \vartheta],\ l \in R^n \tag{5.6}$$

$\operatorname{dom}\rho(\alpha) = \{r \in R^n\colon \rho(r, \alpha) < +\infty\}$ is a compact set in R^n for all $\alpha \in A$ by the Lipschitz condition for φ_i.

The VMF $(t, x) \to \mathrm{pm}(t, x)$: $T \times R^n \to 2^{R^m}$, $\mathrm{pm} \in \mathrm{SC}$ whose epigraph is the programmed absorption set Z will be called programmed maximin vector multivalued function (PM VMF), i.e.,

$$\mathrm{pm}(t, x) = \{s^\circ \in Z(t, x) : s \nleqslant s^\circ \text{ for all } s \in Z(t, x) \setminus \{s^\circ\}\} \tag{5.7}$$

$$Z(t, x) = \{s \in R^m\colon (t, x, s) \in Z\} \tag{5.8}$$

6. Regularity conditions for PM VMF. The programmed absorption set Z in general is not identical with the positional absorption set W_u in problem (2.1), (2.2). We have $W_u \in Z$, because Z is v-stable /3/. If the set Z is moreover u-stable, then we should have the converse inclusion $Z \subseteq W_u$ and therefore the equality $W_u = Z$. But then the OGR VMF $(t, x) \to c(t, x)$ is identical with PM VMF $(t, x) \to \mathrm{pm}(t, x)$, if $(t, x) \to \mathrm{pm}(t, x)$ is u-stable. This case is called regular and the conditions ensuring u-stability of PM VMF are called regularity conditions.

Definition 3 of u-stability of VMF includes its derivative (4.5)-(4.7). The derivative of PM VMF $(t, x) \to \mathrm{pm}(t, x)$ defined by formulas (5.4)-(5.8) may be evaluated by the tools of non-smooth analysis /14/. We give the final expression for this derivative.

The lower derivative of PM VMF $(t, x) \to \mathrm{pm}(t, x)$ at the point (t_*, x_*, s_*), $(t_*, x_*) \in T^\circ \times R^n$, $s_* \in \mathrm{pm}(t_*, x_*)$ with respect to the direction $h \in R^n$ is defined by the relationships

$$\partial_-\, \mathrm{pm}(t_*, x_*, s_*)|(h) = \{d^\circ \in \nabla\, \mathrm{pm}(t_*, x_*, s_*)|(h)\colon d \nleqslant d^\circ \text{ for all } d \in \nabla\, \mathrm{pm}(t_*, x_*, s_*)|(h) \setminus \{d^\circ\}\} \tag{6.1}$$

$$\nabla\, \mathrm{pm}(t_*, x_*, s_*)|(h) = \{d \in R^m : \min_{\alpha\in A(t_*, x_*, s_*)} (\langle \alpha, d\rangle - \max_{l\in L(t_*, x_*, s_*, \alpha)} (\langle l, h\rangle - H(t_*, l))) \geqslant 0\} \tag{6.2}$$

Here

$$A(t_*, x_*, s_*) = \{\alpha_* \in A : \alpha_* = \operatorname*{argmin}_{\alpha\in A} (\langle \alpha, s_*\rangle - g(t_*, x_*, \alpha))\} \tag{6.3}$$

$$L(t_*, x_*, s_*, \alpha) = \Big\{l_* \in \operatorname{dom}\rho(\alpha) : l_* = \operatorname*{argmax}_{l\in \operatorname{dom}\rho(\alpha)} (\langle l, x_*\rangle + \int_{t_*}^\vartheta H(\tau, l)\,d\tau - \rho(l, \alpha))\Big\},\quad \alpha \in A(t_*, x_*, s_*) \tag{6.4}$$

Using equalities (6.1)-(6.4) to rewrite Definition 3, which expresses the u-stability property in infinitesimal form, we obtain the following regularity condition for PM VMF $(t, x) \to \mathrm{pm}\,(t, x)$.

Regularity condition 1. PM VMF $(t, x) \to \mathrm{pm}\,(t, x)$ is identical with the OGR VMF $(t, x) \to c\,(t, x)$ if and only if for any positions $(t_*, x_*) \in T^\circ \times R^n$, any values $s_* \in \mathrm{pm}\,(t_*, x_*)$, and any controls $v_* \in Q$ there are a control $u_* \in P$ and a vector $d \in R^m$ that satisfy the following conditions:

$$\min_{\alpha \in A(t_*, x_*, s_*)} (\langle \alpha, d\rangle - \max_{l \in L(t_*, x_*, s_*, \alpha)} (\langle l, B(t_*) u_*\rangle - \min_{u \in P} \langle l, B(t_*) u\rangle + \langle l, C(t_*) v_*\rangle - \max_{v \in Q} \langle l, C(t_*) v\rangle)) \geqslant 0 \tag{6.5}$$

$$d \leqslant 0 \tag{6.6}$$

Remark 3. Relationships (6.5), (6.6) are analogous to the regularity conditions of the programmed maximin function in the differential game with a scalar criterion /5, 6/.

The regularity condition for PM VMF $(t, x) \to \mathrm{pm}\,(t, x)$ may be restated in terms of conjugate variables.

Regularity condition 2. PM VMF $(t, x) \to \mathrm{pm}\,(t, x)$ is identical with OGR VMF $(t, x) \to c\,(t, x)$ if and only if for any positions $(t_*, x_*) \in T^\circ \times R^n$ and any values $s_* \in \mathrm{pm}\,(t_*, x_*)$ we have

$$\sum_{k=1}^{n+1} \gamma_k H(t_*, l_k) \geqslant H\left(t_*, \sum_{k=1}^{n+1} \gamma_k l_k\right) \tag{6.7}$$

$$l_k \in \bigcup_{\alpha \in A(t_*, x_*, s_*)} L(t_*, x_*, s_*, \alpha), \gamma_k \geqslant 0, k = 1, \ldots, n+1, \sum_{k=1}^{n+1} \gamma_k = 1 \tag{6.8}$$

Proof. Consider the multivalued mapping $t \to Z(t)$, where

$$Z(t) = \{(x, s) \in R^n \times R^m : (t, x, s) \in Z\}$$

Let $(t_*, x_*) \in T^\circ \times R^n$, $s_* \in \mathrm{pm}\,(t_*, x_*)$. For the derivative of a multivalued mapping $t \to Z(t)$ we have at the point (t_*, x_*, s_*)

$$DZ(t_*, x_*, s_*) = \{(\xi, d) \in R^n \times R^m : \min_{\alpha \in A(t_*, x_*, s_*)} (\langle \alpha, d\rangle - \max_{l \in L(t_*, x_*, s_*, \alpha)} (\langle l, \xi\rangle - H(t_*, l))) \geqslant 0\} \tag{6.9}$$

By the results of /9/, the u-stability condition of a multivalued mapping $t \to Z(t)$ may be rewritten in the following form: for any positions $(t_*, x_*) \in T^\circ \times R^n$, any values $s_* \in \mathrm{pm}\,(t_*, x_*)$, and any vector $v_* \in Q$, we have

$$DZ(t_*, x_*, s_*) \cap \mathrm{FR}\,(t_*, x_*, s_*, v_*) \neq \varnothing \tag{6.10}$$

$$\mathrm{FR}\,(t_*, x_*, s_*, v_*) = \{(B(t_*) u + C(t_*) v_*, 0) : u \in P, 0 \in R^m\}$$

Since $DZ(t_*, x_*, s_*)$ in this case is a convex closed set and $\mathrm{FR}\,(t_*, x_*, s_*, v_*)$ is a convex compactum, then by the separability theorem for convex sets /15/ relationship (6.10) may be written in the form

$$\sup_{(\xi, d) \in DZ(t_*, x_*, s_*)} (\langle l, \xi\rangle - \langle \alpha, d\rangle) \geqslant H(t_*, l) \tag{6.11}$$

for all $(t_*, x_*) \in T^\circ \times R^n$, $s_* \in \mathrm{pm}\,(t_*, x_*)$, $(l, \alpha) \in R^n \times A$.

Using (6.9), we obtain /15/

$$\sup_{(\xi, d) \in DZ(t_*, x_*, s_*)} (\langle l, \xi\rangle - \langle \alpha, d\rangle) = \begin{cases} +\infty, & \text{if } (l, \alpha) \notin \mathrm{co}\,\{\mathrm{AL}\} \\ \sum_{k=1}^{n+m+1} \beta_k H(t_*, l_k), & \text{if } (l, \alpha) \in \mathrm{co}\,\{\mathrm{AL}\} \end{cases} \tag{6.12}$$

Here

$$\mathrm{co}\,\{\mathrm{AL}\} = \Big\{(l^\circ, \alpha^\circ) \in R^n \times A : (l^\circ, \alpha^\circ) = \sum_{k=1}^{n+m+1} \beta_k^\circ (l_k^\circ, \alpha_k^\circ),\ \alpha_k^\circ \in A(t_*, x_*, s_*),$$
$$l_k^\circ \in L(t_*, x_*, s_*, \alpha^\circ),\ \beta_k^\circ \geqslant 0,\ k = 1, \ldots, n+m+1,\ \sum_{k=1}^{n+m+1} \beta_k^\circ = 1\Big\} \tag{6.13}$$

the coefficients β_k and the vectors $l_k,\ k = 1, \ldots, n + m + 1$, correspond to the vector $(l, \alpha) \in R^n \times A$ in the expansion (6.13).

Using (6.12) and (6.13), we can rewrite (6.11) in the form

$$\sum_{k=1}^{n+m+1} \beta_k H(t_*, l_k) \geqslant H\left(t_*, \sum_{k=1}^{n+m+1} \beta_k l_k\right) \tag{6.14}$$

$$l_k \in \bigcup_{\alpha \in A(t_*, x_*, s_*)} L(t_*, x_*, s_*, \alpha),\ \beta_k \geqslant 0,\ k = 1, \ldots, n + m + 1,\ \sum_{k=1}^{n+m+1} \beta_k = 1$$

By Caratheodory's theorem /15/, we need retain in (6.14) only the sums with index varying from 1 to $n + 1$. We thus finally obtain the regularity condition (6.7), (6.8).

Remark 4. Relationships (6.7), (6.8) are similar in form to the regularity condition of the PM function in a differential game with a scalar criterion /4/.

Remark 5. In this paper, the regularity conditions (6.5), (6.6) and (6.7), (6.8) were obtained using differential inequalities. However, their final form does not contain infinitesimal constructions.

*7. **Example***. Consider a second-order linear system

$$z_1^{\cdot} = z_2 + v,\ t \in [0, 1],\ z = (z_1, z_2) \in R^2,\ |u| \leqslant 1,\ |v| \leqslant 1 \qquad z_2^{\cdot} = u \tag{7.1}$$

The vector payoff function is defined by

$$\sigma(z) = (\sigma_1(z), \sigma_2(z)),\ \sigma_1(z) = |z_1|,\ \sigma_2(z) = |z_2| \tag{7.2}$$

Note that system (7.1) with the scalar criterion $\max\{|z_1|, |z_2|\}$ was studied in /16/.

Making the traditional change of variables $x = \Phi(t, \vartheta) z$ ($\Phi(t, \vartheta)$ is the fundamental matrix of the homogeneous system), we obtain the system

$$x_1^{\cdot} = (1 - t) u + v,\ t \in [0, 1],\ x = (x_1, x_2) \in R^2,\ |u| \leqslant 1,\ |v| \leqslant 1 \qquad x_2^{\cdot} = u \tag{7.3}$$

with the vector payoff functional

$$\varphi(x) = (\varphi_1(x), \varphi_2(x)),\ \varphi_1(x) = |x_1|,\ \varphi_2(x) = |x_2| \tag{7.4}$$

A complete analysis of this example, including evaluating the PM VMF $(t, x) \to \mathrm{pm}(t, x)$ in all positions $(t_*, x_*) \in [0, 1] \times R^2$ and checking the relationships (6.5), (6.6) or (6.7), (6.8), as required by the regularity conditions, is much too complicated. We will therefore only consider the case $t_* = t\ (t \in [0, 1]),\ x_* = 0$.

We have

$$A = \{\alpha \in R^2 : \alpha_k \geqslant 0,\ k = 1, 2,\ \alpha_1 + \alpha_2 = 1\} \tag{7.5}$$

$$\rho(l, \alpha) = \sup_{x \in R^2} (l_1 x_1 + l_2 x_2 - \alpha_1 |x_1| - \alpha_2 |x_2|) = \begin{cases} 0, & \text{if } |l_1| \leqslant \alpha_1 \text{ and } |l_2| \leqslant \alpha_2 \\ +\infty, & \text{if } |l_1| > \alpha_1 \text{ or } |l_2| > \alpha_2 \end{cases}$$

$$g(t, x, \alpha) = \max_{|l_k| \leqslant \alpha_k,\ k=1,2} \left(l_1 x_1 + l_2 x_2 + \int_t^1 |l_1|\, d\tau - L(t)\right)$$

$$L(t) = \int_t^1 |l_1(1 - \tau) + l_2|\, d\tau$$

In the position $(t_*, x_*) = (t, 0),\ t \in [0, 1]$, we obtain

$$g(t, 0, \alpha) = \max_{|l_k| \leqslant \alpha_k,\ k=1,2} ((1 - t) l_1 - L(t))$$

We can show that the maximum is reached for $l_1 = \alpha_1$. Therefore

$$g(t, 0, \alpha) = \max_{|l_2| \leqslant \alpha_2} \left((1 - t)\alpha_1 - \int_t^1 |\alpha_1(1 - \tau) + l_2|\, d\tau\right) = \begin{cases} \alpha_1(1 - t) - (\alpha_2^2 + (\alpha_1(1 - t) - \alpha_2)^2)(2\alpha_1)^{-1}, & \text{if } 2\alpha_2 \leqslant \alpha_1(1 - t) \\ \alpha_1(1 - t) - {}^1/_4\alpha_1(1 - t)^2, & \text{if } 2\alpha_2 \geqslant \alpha_1(1 - t) \end{cases} \tag{7.6}$$

From (5.4), (5.8) and (7.6) we obtain after some reduction

$$Z(t, 0) = \{s \in R^2 : \min\{\eta_1(s), \eta_2(s), \eta_3(s)\} \geqslant 0\} \tag{7.7}$$

$$\eta_1(s) = s_2,\ \eta_2(s) = s_1 - (1 - t) - \nu(t)$$

$$\eta_3(s) = \begin{cases} s_2 - 2h(t) + 2(s_1 - s_2 + 1 + \nu(t))^{1/2}, & \text{if } \nu(t) \leqslant s_1 - s_2 \leqslant \mu(t) \\ -\infty, & \text{if } \nu(t) > s_1 - s_2 \text{ or } s_1 - s_2 > \mu(t) \end{cases}$$

$$h(t) = 1 + {}^1/_2(1-t),\ \nu(t) = -{}^1/_2(1-t)^2,\ \mu(t) = (1-t) - {}^1/_4(1-t)^2$$

From (7.7) we see that PM VMF $(t, x) \to \mathrm{pm}(t, x)$ at the point $(t_*, x_*) = (t, 0)$, $t \in [0, 1]$, is the set consisting of a piece of the parabola

$$\mathrm{pm}(t, 0) = \{s \in R^2 : s_1 - {}^1/_4(s_2 - (1-t))^2 - (1-t) + {}^1/_2(1-t)^2 = 0,\ 0 \leqslant s_2 \leqslant 1-t\}$$

Let us check the regularity conditions (6.7), (6.8) in the positions $(t_*, x_*) = (t, 0)$, $t \in [0, 1)$. Recall that the Hamiltonian of system (7.3) is defined by the relationship

$$H(t, l) = \max_{|v| \leqslant 1} l_1 v + \min_{|u| \leqslant 1} (l_1(1-t) + l_2) u = |l_1| - |l_1(1-t) + l_2|$$

Let $s = (s_1, s_2) \in \mathrm{pm}(t, 0)$, $t \in [0, 1)$. We will consider two cases: 1) $0 < s_2 \leqslant 1-t$; 2) $s_2 = 0$. In the first case, we have

$$A(t, 0, s) = \{\alpha = (\alpha_1, \alpha_2) : \alpha_1 = 2(2 + (1-t) - s_2)^{-1},\ \alpha_2 = ((1-t) - s_2)(2 + (1-t) - s_2)^{-1}\}$$

$$L(t, 0, s, \alpha) = \{a = (a_1, a_2),\ b = (b_1, b_2) : a_1 = \alpha_1,\ a_2 = -\alpha_2,\ b = -a\}$$

We obtain

$$H(t, a) = H(t, b) = |\alpha_1| - |\alpha_1(1-t) - \alpha_2| = (2 - (1-t) - s_2) \cdot (2 + (1-t) - s_2)^{-1}$$

$$H(t, \gamma a + (1-\gamma) b) = |2\gamma - 1|(2 - (1-t) - s_2)(2 + (1-t) - s_2)^{-1}$$

$$0 \leqslant \gamma \leqslant 1,\ 0 < s_2 \leqslant 1-t$$

Since

$$\gamma H(t, a) + (1-\gamma) H(t, b) = H(t, a) = (2 - (1-t) - s_2)(2 + (1-t) - s_2)^{-1} \geqslant |2\gamma - 1|(2 - (1-t) - s_2)(2 + (1-t) - s_2)^{-1} = H(t, \gamma a + (1-\gamma) b),\ 0 \leqslant \gamma \leqslant 1$$

conditions (6.7) and (6.8) are satisfied.

In the second case, we have

$$A(t, 0, s) = \{\varkappa = (\varkappa_1, \varkappa_2),\ \lambda = (\lambda_1, \lambda_2) : \varkappa_1 = 2(2 + (1-t))^{-1},\ \varkappa_2 = (1-t)(2 + (1-t))^{-1},\ \lambda_1 = 0,\ \lambda_2 = 1\}$$

$$L(t, 0, s, \varkappa) = \{a = (a_1, a_2),\ b = (b_1, b_2) : a_1 = \varkappa_1,\ a_2 = -\varkappa_2,\ b = -a\}$$

$$L(t, 0, s, \lambda) = \{c = (c_1, c_2) : c_1 = 0,\ c_2 = 0\}$$

$$H(t, a) = H(t, b) = |\varkappa_1| - |\varkappa_1(1-t) - \varkappa_2| = (2 - (1-t))(2 + (1-t))^{-1},\quad H(t, c) = 0$$

$$H(t, \gamma_1 a + \gamma_2 b + \gamma_3 c) = |\gamma_1 - \gamma_2|(2 - (1-t))(2 + (1-t))^{-1}$$

$$\gamma_k \geqslant 0,\ k = 1, 2, 3,\ \gamma_1 + \gamma_2 + \gamma_3 = 1$$

Since

$$\gamma_1 H(t, a) + \gamma_2 H(t, b) + \gamma_3 H(t, c) = (\gamma_1 + \gamma_2) H(t, a) = (\gamma_1 + \gamma_2)(2 - (1-t))(2 + (1-t))^{-1} \geqslant |\gamma_1 - \gamma_2|(2 - (1-t))(2 + (1-t))^{-1} = H(t, \gamma_1 a + \gamma_2 b + \gamma_3 c),\ \gamma_k \geqslant 0,\ k = 1, 2, 3,\ \gamma_1 + \gamma_2 + \gamma_3 = 1$$

conditions (6.7) and (6.8) are satisfied.

We can similarly evaluate PM VMF $(t, x) \to \mathrm{pm}(t, x)$ in other positions $(t_*, x_*) \in [0, 1] \times R^2$ and check the regularity conditions (6.7) and (6.8).

In this example, the PM VMF $(t, x) \to \mathrm{pm}(t, x)$ is identical with the OGR VMF $(t, x) \to c(t, x)$.

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Translated by Z.L.

J. Appl. Maths Mechs, Vol. 55, No. 2, pp. 176-180, 1991
Printed in Great Britain

0021-8928/91 $15.00+0.00

ACCESSORY PARAMETERS IN CIRCULAR QUADRANGLES*

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The problem of the conformal mapping of a circular polygon in a half-plane (a half-strip, a rectangle, a circle, etc.) is of considerable importance, e.g. in the theory of groundwater motion. The question of accessory (redundant) parameters that may arise in this case is not trivial and deserves special analysis. Some special cases of such problems are considered in this paper.

1. A circular triangle in a plane is completely defined by the position of its three vertices and the three angles at the vertices. For a circular quadrangle, the specification of three of its vertices and the four angles does not completely define the quadrangle, and the fourth vertex may have an infinite set of positions /1, p.306/. Let us consider this problem in more detail for the case of a circular rectangle (Fig.1) with given sides $A_2A_3 = a$ and $A_1A_2 = b$. Draw two families of auxiliary circles tangent to the segments A_1A_4 and A_3A_4 at the points A_1 and A_3, respectively. An infinite set of these circles intersects at a right angle (points A and A'). We will show that the family of such points $A(x, y)$ is a circle through the vertices of the rectangle $A_1A_2A_3A_4$ (Fig.1).

Let $(0, b_1)$ and $(a_1, 0)$ be the centres of two circles through the point A. Then the equations of these circles are

$$x^2 + (y - b_1)^2 = (b_1 - b)^2,\ (x - a_1)^2 + y^2 = (a_1 - a)^2 \tag{1.1}$$

The expressions for the slope of the tangents at the point A

$$y_1' = -x/(y - b_1),\quad y_2' = -(x - a_1)/y$$

* *Prikl.Matem.Mekhan.*,55,2,222-227,1991